# A GENERALIZATION OF GRÜNBAUM'S INEQUALITY

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Abstract. Grünbaum's inequality gives sharp bounds between the volume of a convex body and its part cut off by a hyperplane through the centroid of the body. We provide a generalization of this inequality for hyperplanes that do not necessarily contain the centroid. As an application, we obtain a sharp inequality that compares sections of a convex body to the maximal section parallel to it.

## 1. Introduction

A *convex body* $K$ is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. As usual, we write $\langle\cdot,\cdot\rangle$ for the Euclidean inner product. We also denote by $S^{n-1}$ the unit sphere in $\mathbb{R}^n$. The *centroid* (also called the *center of mass*, or *barycenter*) of $K$ is the point

$$g(K) = \frac{1}{|K|}\int_K x\,dx.$$

Here and throughout the paper, $|A|_k$ denotes the $k$-dimensional Lebesgue measure (volume) of a $k$-dimensional measurable set $A$. If the dimension of the set is understood, then we will omit the subscript. An inequality of Grünbaum [5] states if $K \subset \mathbb{R}^n$ is a convex body with centroid at the origin then

$$\left(\frac{n}{n+1}\right)^n \le \frac{|K\cap\xi^+|}{|K|} \le 1-\left(\frac{n}{n+1}\right)^n, \quad \text{for all } \xi\in S^{n-1}. \tag{1}$$

Here $\xi^+ = \{x\in\mathbb{R}^n : \langle x,\xi\rangle \ge 0\}$. The bounds in (1) are sharp and equality occurs in the lower bound when, for example, $K$ is the cone

$$K = \operatorname{conv}\left(\frac{-1}{n+1}\xi + B_2^{n-1}, \frac{n}{n+1}\xi\right). \tag{2}$$

Similarly, equality occurs in the upper bound when, for example, $K$ is the cone

$$K = \operatorname{conv}\left(\frac{n}{n+1}\xi + B_2^{n-1}, \frac{1}{n+1}\xi\right),$$

where we denote by $B_2^{n-1}$ the closed unit $(n-1)$-dimensional Euclidean ball in $\xi^\perp = \{x\in\mathbb{R}^n : \langle x,\xi\rangle = 0\}$. In this paper, all sets will be closed. For recent advancements in Grünbaum-type inequalities for sections and projections of convex bodies see [3], [8], [9], [12].

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In light of (1), the goal of this paper is to establish a similar result with hyperplanes that do not necessarily contain the centroid. We ask the following question: Are there positive constants $C_1$ and $C_2$ such that

$$C_1 \leq \frac{|K \cap H_\alpha^+|}{|K|} \leq C_2, \tag{3}$$

for every convex body with centroid at the origin, $\alpha \in (-1, n)$, and $\xi \in S^{n-1}$? Here $H_\alpha^+$ is the halfspace

$$H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq \alpha h_K(-\xi)\},$$

and $h_K$ is the support function for $K$ (see Section 2 for the precise definition). We give an affirmative answer to this question. The two constants $C_1$ and $C_2$ depend only on $\alpha$ and $n$, i.e., $C_1 = C_1(\alpha, n)$, $C_2 = C_2(\alpha, n)$. Both bounds are sharp, and the exact values of $C_1(\alpha, n)$ and $C_2(\alpha, n)$ are presented in Theorem 4, which also discusses the equality cases. The case $n = 2$ for (3) was obtained earlier in [11], where it was used to prove a discrete version of Grünbaum's inequality. It is important to note that in (1), one bound automatically determines the other bound. On the other hand, the bounds in (3) need to be shown separately.

As an application of (3) we obtain a generalization of the following result of Makai and Martini [7]; see also [2]. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, then

$$\left|K \cap \xi^\perp\right| \geq \left(\frac{n}{n+1}\right)^{n-1} \max_{t \in \mathbb{R}} \left|K \cap \left(\xi^\perp + t\xi\right)\right|, \quad \text{for all } \xi \in S^{n-1}. \tag{4}$$

The bound is sharp, and equality holds again if $K$ is a cone as in (2). In this paper, we establish an analogue of the inequality above for sections that do not necessarily pass through the centroid. Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin, $\alpha \in (-1, n)$, and $\xi \in S^{n-1}$. Consider the hyperplane

$$H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = \alpha h_K(-\xi)\}.$$

Then

$$|K \cap H_\alpha| \geq D \max_{t \in \mathbb{R}} \left|K \cap \left(\xi^\perp + t\xi\right)\right|,$$

where $D = D(\alpha, n)$ is a constant depending on only $\alpha$ and $n$. The inequality is sharp, and the exact value of $D(\alpha, n)$ is discussed in Theorem 5, along with equality cases.

## 2. Preliminaries

The *support function* $h_K : \mathbb{R}^n \longrightarrow \mathbb{R}$ for a convex body $K \subset \mathbb{R}^n$ is

$$h_K(\xi) = \max\{\langle x, \xi \rangle : x \in K\}.$$

If $\xi \in S^{n-1}$ then $h_K(\xi)$ gives the signed distance from the origin to the supporting hyperplane for $K$ in the direction $\xi$. A result of Minkowski and Randon [1, p. 58] states if $K \subset \mathbb{R}^n$ is a convex body with centroid at the origin and $\xi \in S^{n-1}$, then

$$\frac{1}{n} h_K(\xi) \leq h_K(-\xi) \leq n h_K(\xi). \tag{5}$$

Note that the choice of bounds for $\alpha$ in Theorems 4 and 5 is a result of (5).

Let $\xi \in S^{n-1}$. The *parallel section function* $A_{K,\xi} : \mathbb{R} \longrightarrow \mathbb{R}$ for a convex body $K$ is

$$A_{K,\xi}(t) = \left|K \cap \left(\xi^{\perp} + t\xi\right)\right|.$$

**Lemma 1.** *Let $K \subset \mathbb{R}^n$ be a convex body. Then $A_{K,\xi}^{1/(n-1)}$ is concave on its support, for every $\xi \in S^{n-1}$.*

For the proof of Lemma 1, refer to [6, p. 18].

Let $\xi \in S^{n-1}$. The *volume cut-off function* $V_{K,\xi} : \mathbb{R} \longrightarrow \mathbb{R}$ for a convex body $K \subset \mathbb{R}^n$ is

$$V_{K,\xi}(t) = \int_t^{\infty} A_{K,\xi}(s)\, ds.$$

The following result is also well-known, but we include a proof for completeness.

**Lemma 2.** *Let $K \subset \mathbb{R}^n$ be a convex body. Then $V_{K,\xi}^{1/n}$ is concave on its support, for every $\xi \in S^{n-1}$.*

*Proof.* Let $\lambda \in [0,1]$ and $t_1, t_2 \in \operatorname{supp}(V_{K,\xi})$. Note that

$$\lambda\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_1\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_2\}\Big) \subset \Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \lambda t_1 + (1-\lambda)t_2\}\Big).$$

This, together with the Brunn-Minkowski inequality, implies that

$$\begin{aligned}
&\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \lambda t_1 + (1-\lambda)t_2\}\Big|^{1/n} \\
&\quad\geq \Big|\lambda\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_1\}\Big) + (1-\lambda)\Big(K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_2\}\Big)\Big|^{1/n} \\
&\quad\geq \lambda\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_1\}\Big|^{1/n} + (1-\lambda)\Big|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq t_2\}\Big|^{1/n},
\end{aligned}$$

which proves the result. □

Let $K \subset \mathbb{R}^n$ be a convex body and $\xi \in S^{n-1}$. The *Schwarz symmetral* of $K$ with respect to $\xi$ is the convex body $\mathcal{S}_\xi K$ such that for all $t \in [-h_K(-\xi), h_K(\xi)]$, the set $\mathcal{S}_\xi K \cap \left(\xi^{\perp} + t\xi\right)$ is an $(n-1)$-dimensional Euclidean ball centered at $t\xi$ and $A_{K,\xi}(t) = A_{(\mathcal{S}_\xi K),\xi}(t)$. By construction we obtain

$$h_K(\pm\xi) = h_{\mathcal{S}_\xi K}(\pm\xi) \quad \text{and} \quad V_{K,\xi}(t) = V_{(\mathcal{S}_\xi K),\xi}(t), \tag{6}$$

for all $t \in \mathbb{R}$. Note that the centroid of $\mathcal{S}_\xi K$ lies on $\ell = \{t\xi : t \in \mathbb{R}\}$ due to the rotational symmetry of $\mathcal{S}_\xi K$ about $\ell$. See [4, p. 62] for more information on Schwarz symmetrizations.

## 3. Main Results

Before proving our main result, we will provide a simple Remark that we will apply throughout the rest of the paper.

**Remark 3.** *Let $K \subset \mathbb{R}^n$ be a convex body with centroid at the origin. Let $\alpha \in (-1, n)$ and $\xi \in S^{n-1}$. Denote $\overline{K} = K + h_K(-\xi)\xi$ and consider the two* *halfspaces* $H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq \alpha h_K(-\xi)\}$ *and* $\overline{H}_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq (\alpha+1)\left\langle g(\overline{K}), \xi \right\rangle\}$. *Then*

$$\left|K \cap H_\alpha^+\right| = \left|\overline{K} \cap \overline{H}_\alpha^+\right|.$$

*Proof.*

$$g(\overline{K}) = \frac{1}{|\overline{K}|}\int_{\overline{K}} x\,dx = \frac{1}{|K|}\int_{K+h_K(-\xi)\xi} x\,dx$$
$$= \frac{1}{|K|}\int_K x + h_K(-\xi)\xi\,dx = h_K(-\xi)\xi,$$

and the result follows. □

Analogous statements to Remark 3 also hold when $\geq$ is replaced with $\leq$ or $=$. We will now prove our main result.

**Theorem 4.** *Let* $K \subset \mathbb{R}^n$ *be a convex body with centroid at the origin. Let* $\alpha \in (-1, n)$ *and* $\xi \in S^{n-1}$. *Consider the halfspace*

$$H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \geq \alpha h_K(-\xi)\}.$$

*Then*

$$C_1(\alpha, n) \leq \frac{|K \cap H_\alpha^+|}{|K|} \leq C_2(\alpha, n).$$

*where*

$$C_1(\alpha,n) = \begin{cases} \left(\frac{n-\alpha}{n+1}\right)^n & \alpha \in (-1,0], \\ \left(\frac{n}{n+1}\right)^n (\alpha+1)^{n-1}(1-\alpha n) & \alpha \in (0, 1/n), \\ 0 & \alpha \in [1/n, n), \end{cases}$$

*and*

$$C_2(\alpha,n) = \begin{cases} 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n & \alpha \in (-1,0], \\ c(\alpha,n) & \alpha \in (0,n). \end{cases}$$

$c(\alpha, n)$ *is a constant depending on only* $\alpha$ *and* $n$. *Determining the explicit value of* $c(\alpha, n)$ *involves finding the roots of a high-degree rational function. The lower bounds and upper bounds are sharp, and equality cases are discussed in the proof below.*

*Proof.* Given $K$ as written above, consider the Schwarz symmetral $\mathcal{S}_\xi K$. Using the observations in (6) and Fubini's theorem we can conclude that the centroid of $\mathcal{S}_\xi K$ is at the origin and that $|K \cap H_\alpha^+| = |(\mathcal{S}_\xi K) \cap H_\alpha^+|$ for all $\alpha \in (-1, n)$. Therefore we will prove the result with $\mathcal{S}_\xi K$, which we will denote by $K$ for brevity. By Remark 3, it suffices to find bounds for $|\overline{K} \cap \overline{H}_\alpha^+|$, and after further abuse of notation, we will write $K$ for $\overline{K}$ and $H_\alpha^+$ for $\overline{H}_\alpha^+$. We will also write $H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = (\alpha+1)\langle g(K), \xi \rangle\}$ and $H_\alpha^- = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq (\alpha+1)\langle g(K), \xi \rangle\}$. Let us remark that $\xi^\perp$ is now a supporting hyperplane of $K$ and $0 \in \partial K$.

Let us first consider the case $\alpha \in (-1, 0]$. We will obtain the upper bound. Observe that

$$\left|K \cap H_\alpha^-\right| = |K| - \left|K \cap H_\alpha^+\right|.$$

Denote by $K/(\alpha+1)$ the dilation of $K$ by a factor of $1/(\alpha+1) > 1$, and also write $H_\alpha^-/(\alpha+1) = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \le \langle g(K), \xi\rangle\}$. Since $0 \in K$, we obtain $K \subset K/(\alpha+1)$ and thus

$$\begin{aligned}\left|K \cap H_\alpha^-\right| &= (\alpha+1)^n \left|\frac{1}{\alpha+1} K \cap \frac{1}{\alpha+1} H_\alpha^-\right| \\ &\ge (\alpha+1)^n \Big| K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \le \langle g(K), \xi\rangle\}\Big| \\ &= (\alpha+1)^n \Big| \big(K - g(K)\big) \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \le 0\}\Big| \ge (\alpha+1)^n \left(\frac{n}{n+1}\right)^n |K|,\end{aligned}$$

where we used Grünbaum's inequality (1). Therefore, for $\alpha \in (-1, 0]$ :

$$\frac{|K \cap H_\alpha^+|}{|K|} \le 1 - \left(\frac{n(\alpha+1)}{n+1}\right)^n,$$

as desired.

We will now obtain the lower bound for $\alpha \in (-1, 0]$. By Lemma 2, $V_{K,\xi}^{1/n}$ is concave on its support. Hence,

$$\begin{aligned}\left|K \cap H_\alpha^+\right|^{1/n} &= V_{K,\xi}{}^{1/n}\big((\alpha+1)\langle g(K), \xi\rangle\big) = V_{K,\xi}{}^{1/n}\big(-\alpha \cdot 0 + (\alpha+1)\langle g(K), \xi\rangle\big) \\ &\ge -\alpha V_{K,\xi}{}^{1/n}(0) + (\alpha+1) V_{K,\xi}{}^{1/n}\big(\langle g(K), \xi\rangle\big).\end{aligned}$$

Using Grünbaum's inequality and the observation that $V_{K,\xi}(0) = |K|$, we have

$$\left|K \cap H_\alpha^+\right|^{1/n} \ge -\alpha |K|^{1/n} + (\alpha+1)\left(\frac{n}{n+1}\right)|K|^{1/n},$$

which implies for $\alpha \in (-1, 0]$ :

$$\left(\frac{n-\alpha}{n+1}\right)^n \le \frac{|K \cap H_\alpha^+|}{|K|}.$$

Thus, we have shown the bounds for $\alpha \in (-1, 0]$.

We will now investigate the case $\alpha \in (0, n)$. We prove the upper bound first. Let $B_2^{n-1}$ be the unit $(n-1)$-dimensional Euclidean ball in $\xi^\perp$. By continuity we can find $r_1 \ge 0$ such that

$$K \cap \xi^\perp \subset r_1 B_2^{n-1} \quad \text{and} \quad \left|\operatorname{conv}(r_1 B_2^{n-1}, K \cap H_\alpha)\right| = \left|K \cap H_\alpha^-\right|.$$

Denote $L^- = \operatorname{conv}(r_1 B_2^{n-1}, K \cap H_\alpha)$. Then again by continuity, there are $r_2 \ge 0$ and $\mu$ with $(\alpha+1)\langle g(K), \xi\rangle < \mu < h_K(\xi)$ such that

$$\left|\operatorname{conv}(K \cap H_\alpha, r_2 B_2^{n-1} + \mu\xi)\right| = \left|K \cap H_\alpha^+\right|$$

and

$$L^- \cup \operatorname{conv}(K \cap H_\alpha, r_2 B_2^{n-1} + \mu\xi) = \operatorname{conv}(r_1 B_2^{n-1}, r_2 B_2^{n-1} + \mu\xi).$$

Denote $L^+ = \operatorname{conv}(K \cap H_\alpha, r_2 B_2^{n-1} + \mu\xi)$. Then $L = L^- \cup L^+$ is a convex body whose sections parallel to $\xi^\perp$ are Euclidean balls; see Figure 1. Note that $\langle g(L^-), \xi\rangle \le \langle g(K \cap H_\alpha^-), \xi\rangle$ and $\langle g(L^+), \xi\rangle \le \langle g(K \cap H_\alpha^+), \xi\rangle$, and thus

$$\langle g(L), \xi\rangle \le \langle g(K), \xi\rangle.$$

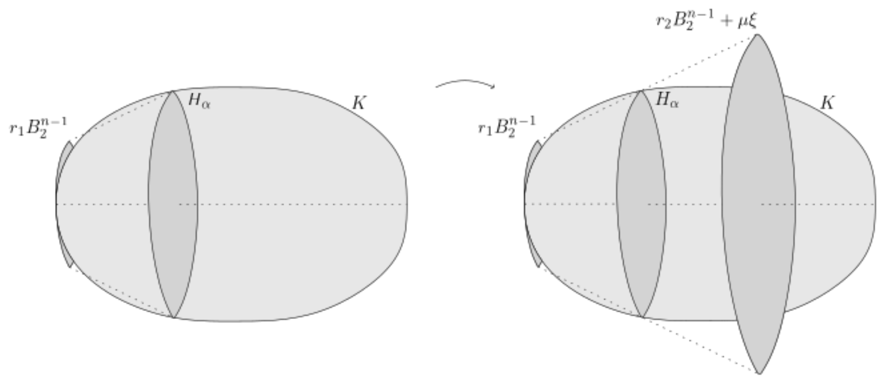


FIGURE 1. Constructing $r_1B_2^{n-1}$ and $r_2B_2^{n-1}+\mu\xi$.

By construction, we have $|K| = |L|$ and

$$\begin{aligned}|K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\langle g(K), \xi\rangle\}| \\ &= |L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\langle g(K), \xi\rangle\}| \\ &\leq |L \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq (\alpha+1)\langle g(L), \xi\rangle\}|\,.\end{aligned}$$

Hence, it suffices to find an upper estimate for $L$ instead of $K$. After rescaling, we may assume that $h_L(\xi) = 1$ and then for $0 \leq t \leq 1$ we can assume that

$$A_{L,\xi}(t) = (mt+b)^{n-1}, \tag{7}$$

where either $m = 0$ and $b > 0$, or $b \geq 0$ and either (1) $m > 0$ or (2) $m < 0$ and $m + b \geq 0$. We will focus on the case $m \neq 0$ first, and address the case $m = 0$ later. Then by Fubini's theorem and (7) we obtain

$$|L| = \int_0^1 A_{L,\xi}(t)\,dt = \frac{(b+m)^n - b^n}{mn}.$$

and similarly we find

$$\langle g(L), \xi\rangle = \frac{1}{|L|}\int_0^1 tA_{L,\xi}(t)\,dt = \frac{b^{n+1} + (mn-b)(b+m)^n}{m(n+1)\big((b+m)^n - b^n\big)}.$$

Denote $f_L = (\alpha+1)\,\langle g(L), \xi\rangle$. Now we can compute

$$\frac{|L \cap H_\alpha^+|}{|L|} = \frac{1}{|L|}\int_{f_L}^1 A_{L,\xi}(t)\,dt = \frac{(b+m)^n - (b+mf_L)^n}{(b+m)^n - b^n}.$$

Denote by $\varphi$ the above equation of $m$ and $b$ for when $m \neq 0$. If $b > 0$ then we have $\varphi(m,b) \xrightarrow{m\to 0} (1-\alpha)/2$, which is readily verified to agree with the case $m = 0$. Making the change of variables $z = b/m$ allows us to write $\varphi$ as a function of $z$. That is, we obtain

$$\varphi(z) = \frac{(z+1)^n - (z+f_L)^n}{(z+1)^n - z^n},$$

where

$$f_L = (\alpha+1)\frac{z^{n+1} + (n-z)(z+1)^n}{(n+1)\big((z+1)^n - z^n\big)},$$

for $z \in (-\infty, -1] \cup [0, \infty)$. For $\alpha \in (0, n)$, it isn't immediately clear whether $\varphi$ has nice properties, so determining $c(\alpha, n)$ becomes an unfeasible task, as solving for $c(\alpha, n)$ involves solving for roots of high-degree rational functions. When $n = 2$ one can explicitly solve for $c(\alpha, n)$ (see [11] for the derivation):

$$c(\alpha, 2) = \begin{cases} \frac{5-3\alpha}{9(\alpha+1)} & \alpha \in (0, 1), \\ \frac{1}{9}(2-\alpha)^2 & \alpha \in [1, 2). \end{cases}$$

Thus, for $\alpha \in (0, n)$ we obtain

$$\frac{|K \cap H_\alpha^+|}{|K|} \leq C_2(\alpha, n) = \sup_{z \in (-\infty, -1] \cup [0, \infty)} \varphi(z),$$

as desired.

We will now obtain the lower bound for $\alpha \in (0, n)$. Note for $\alpha \in [1/n, n)$ if $K$ is the cone

$$K = \operatorname{conv}\left(\frac{-n}{n+1}\xi + B_2^{n-1}, \frac{1}{n+1}\xi\right),$$

then $|K \cap H_\alpha^+| = 0$. Therefore we can not do better than $C_1(\alpha, n) = 0$.

Now assume that $\alpha \in (0, 1/n)$. By continuity, there is $v \geq h_K(\xi)$ such that

$$\left|\operatorname{conv}(K \cap H_\alpha, v\xi)\right| = \left|K \cap H_\alpha^+\right|.$$

Denote $M^+ = \operatorname{conv}(K \cap H_\alpha, v\xi)$. Then again by continuity there are $r$ and $\beta$ with $r > 0$ and $0 < \beta < (\alpha + 1)\langle g(K), \xi\rangle$ such that

$$\operatorname{conv}(rB_2^{n-1} + \beta\xi, M^+) = \operatorname{conv}(rB_2^{n-1} + \beta\xi, v\xi),$$

and

$$\left|\operatorname{conv}(0, rB_2^{n-1} + \beta\xi, K \cap H_\alpha)\right| = \left|K \cap H_\alpha^-\right|.$$

Denote $M^- = \operatorname{conv}(0, rB_2^{n-1} + \beta\xi, K \cap H_\alpha)$. Then $M = M^- \cup M^+$ is a convex body formed by the union of two cones with a common base in $\xi^\perp + \beta\xi$, whose sections parallel to $\xi^\perp$ are Euclidean balls; see Figure 2.

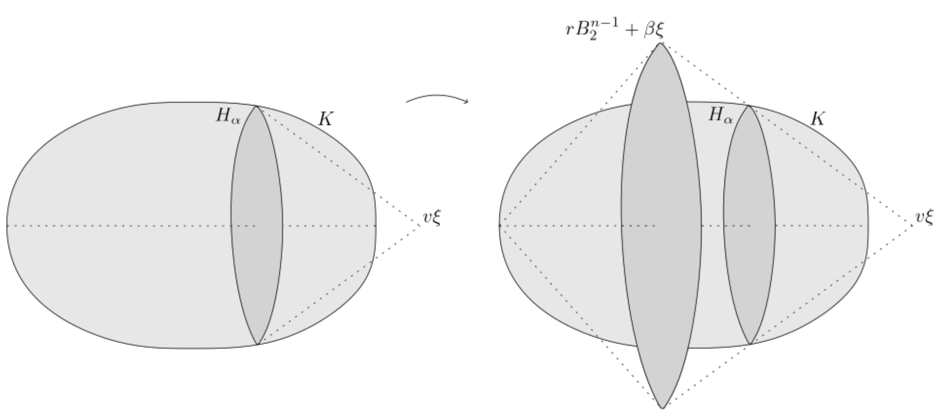


FIGURE 2. Constructing $rB_2^{n-1} + \beta\xi$ and $v\xi$.

Note that $\langle g(M^-),\xi\rangle \geq \langle g(K\cap H_\alpha^-),\xi\rangle$ and $\langle g(M^+),\xi\rangle \geq \langle g(K\cap H_\alpha^+),\xi\rangle$, and thus

$$\langle g(M),\xi\rangle \geq \langle g(K),\xi\rangle\,.$$

As a result we have constructed a convex body $M$ where $|K| = |M|$ and

$$\begin{aligned}|K\cap\{x\in\mathbb{R}^n : x_1 \geq (\alpha+1)\langle g(K),\xi\rangle\}|&\\ &= |M\cap\{x\in\mathbb{R}^n : x_1 \geq (\alpha+1)\langle g(K),\xi\rangle\}|\\ &\geq |M\cap\{x\in\mathbb{R}^n : x_1 \geq (\alpha+1)\langle g(M),\xi\rangle\}|\,.\end{aligned}$$

Hence, it suffices to work with $M$ instead of $K$. Up to rescaling, we may assume that $h_M(\xi) = 1$ and $\left|rB_2^{n-1}\right| = n$. Define

$$M_1 = M\cap\{x\in\mathbb{R}^n \mid \langle x,\xi\rangle \leq \beta\} \quad \text{and} \quad M_2 = M\cap\{x\in\mathbb{R}^n \mid \langle x,\xi\rangle \geq \beta\},$$

to be the cones forming $M$. Since $h_{M_1}(\xi) = \beta$ and $\left|rB_2^{n-1}\right| = n$, an application of Fubini's theorem yields $|M_1| = \beta$ and $|M_2| = 1-\beta$. It is a well-known fact that the centroid of a cone in $\mathbb{R}^n$ divides its height in the ratio $[1:n]$. Hence, we obtain $\langle g(M_1),\xi\rangle = (\beta n)/(n+1)$ and $\langle g(M_2),\xi\rangle = (\beta n+1)/(n+1)$, and thus it follows that

$$\begin{aligned}\langle g(M),\xi\rangle = |M_1|\,\langle g(M_1),\xi\rangle + |M_2|\,\langle g(M_2),\xi\rangle&\\ &= \beta\frac{\beta n}{n+1} + (1-\beta)\frac{\beta n+1}{n+1} = \frac{\beta(n-1)+1}{n+1}.\end{aligned}$$

Denote $f_M = (\alpha+1)\langle g(M),\xi\rangle$. We are interested in computing the volume of the intersection of $M$ with the halfspace $H_\alpha^+ = \{x\in\mathbb{R}^n : \langle x,\xi\rangle \geq f_M\}$. We will consider two cases, first when $0<\beta\leq f_M$, and then when $f_M \leq \beta < 1$. These cases are equivalent to $0<\beta\leq\frac{-\alpha-1}{(n-1)\alpha-2}$ and $\frac{-\alpha-1}{(n-1)\alpha-2}\leq\beta<1$, respectively. In the first case, note that $M\cap H_\alpha^+$ is a cone homothetic to $M_2$ with the homothety coefficient equal to $(1-f_M)/(1-\beta)$. Therefore,

$$\left|M\cap H_\alpha^+\right| = \left(\frac{1-f_M}{1-\beta}\right)^n(1-\beta) = \frac{(1-f_M)^n}{(1-\beta)^{n-1}}.$$

In the second case, $M\cap H_\alpha^-$ is a cone homothetic to $M_1$ with the homothety coefficient equal to $f_M/\beta$. Thus,

$$\left|M\cap H_\alpha^+\right| = 1-\left|M\cap H_\alpha^-\right| = 1-\left(\frac{f_M}{\beta}\right)^n\beta = 1-\frac{f_M^n}{\beta^{n-1}}.$$

Summarizing, $|M\cap H_\alpha^+|$ is equal to the following piecewise function

$$\psi(\beta) = \begin{cases}\frac{(1-f_M)^n}{(1-\beta)^{n-1}}, & 0<\beta\leq\frac{-\alpha-1}{(n-1)\alpha-2},\\ 1-\frac{f_M^n}{\beta^{n-1}}, & \frac{-\alpha-1}{(n-1)\alpha-2}\leq\beta<1.\end{cases}$$

Denote by $\psi$ the above function of $\beta$. Our goal is to find the minimum of $\psi$ on $(0,1)$ when $\alpha\in(0,1/n)$. Calculations show that the derivative of $\psi$ vanishes at $\beta_0 = \big((n+1)\alpha\big)/(\alpha+1) \in (0,\frac{-\alpha-1}{(n-1)\alpha-2})$. Furthermore, $\psi$ is decreasing on $(0,\beta_0)$ and increasing on $(\beta_0,1)$. Thus, the minimum of $\psi$ is

$$\psi(\beta_0) = \left(\frac{n}{n+1}\right)^n(\alpha+1)^{n-1}(1-\alpha n).$$

We will now discuss the equality cases. Recall that in both the upper bound construction and lower bound construction, we performed operations such as the Schwarz symmetrization to transform the sections of $K$ in the direction of $\xi$ into $(n-1)$-dimensional Euclidean balls. We also performed scalings and translations. If we have an equality body $K$ for either bound under these operations, then we can undo these operations to produce a new body whose sections are no longer $(n-1)$-dimensional Euclidean balls but instead $(n-1)$-dimensional convex bodies homothetic to each other.

We will start classifying equality cases for the upper bound. Recall in the upper bound construction, the extremal body $L$ is, up to translation, the convex hull of an $(n-1)$-dimensional convex body $B$ lying parallel to $\xi^\perp$ in $\xi^+$, and a homothetic copy of $B$ lying in $\xi^- = \{x \in \mathbb{R}^n : \langle x, \xi \rangle \leq 0\}$. For $\alpha \in (-1, 0]$, we have equality from the equality conditions of Grünbaum's theorem, in other words $L = \operatorname{conv}(B, v)$ is a cone with its base $B$ being an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in $\xi^+$ and vertex $v$ lying in $\xi^-$. For $\alpha \in (0, n)$, $L$ is still the convex hull of a $(n-1)$-dimensional convex body $B$ and a homothetic copy of $B$, but there is no information on an explicit maximum, so we can not determine all equality cases.

We will now classify equality cases for the lower bound. Recall in the lower bound construction, the extremal body $M$ is, up to translation, the union of two cones which share the same base $B$. For $\alpha \in (-1, 0]$, we have equality in the limit from the equality conditions of Grünbaum's theorem, in other words $M = \operatorname{conv}(B, v)$ is a cone with its base $B$ being an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in $\xi^-$ and vertex $v$ lying in $\xi^+$. For $\alpha \in (0, 1/n)$, recall that we have a minimum for $\psi$ at $\beta_0$. As $\alpha$ increases from 0 towards $1/n$, $\beta_0$ increases from 0 to 1, so $B$ shifts in the direction of $\xi$. When $\alpha \in [1/n, n)$ we have equality in the limiting case for $\alpha \in (0, 1/n)$, $\alpha \longrightarrow 1/n$; see Figure 3. □

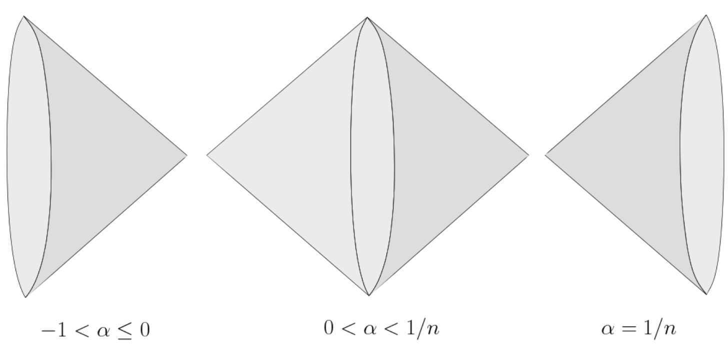


FIGURE 3. Extremizing shapes for the lower bound.

As an application of Theorem 4 we obtain a generalization of the result of Makai and Martini [7] stated in the introduction.

**Theorem 5.** *Let $K$ be a convex body with centroid at the origin. Let $\xi \in S^{n-1}$ and $\alpha \in (-1, n)$. Consider the hyperplane*

$$H_\alpha = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = \alpha h_K(-\xi)\}.$$

*Then*

$$|K \cap H_\alpha| \geq D(\alpha, n) \sup_{t \in \mathbb{R}} \left| K \cap \left(\xi^\perp + t\xi\right) \right|,$$

*where*

$$D(\alpha, n) = \begin{cases} \left(\frac{n(\alpha+1)}{n+1}\right)^{n-1} & \alpha \in (-1, 0], \\ \left(\frac{n-\alpha}{n+1}\right)^{n-1} & \alpha \in (0, 1/n], \\ 0 & \alpha \in (1/n, n). \end{cases}$$

*The bound is sharp and equality cases are discussed in the proof below.*

*Proof.* Note for $\alpha \in (1/n, n)$, if $K$ is the cone

$$K = \operatorname{conv}\left(\frac{-n}{n+1}\xi + B_2^{n-1}, \frac{1}{n+1}\xi\right),$$

then it follows that $|K \cap H_\alpha| = 0$. Therefore for such $\alpha$ we cannot do better than $D(\alpha, n) = 0$.

We will now consider $\alpha \in (-1, 0]$. We can assume that

$$|K \cap H_\alpha| < \sup_{t \in \mathbb{R}} \left| K \cap \left(\xi^\perp + t\xi\right) \right|,$$

otherwise, the theorem follows immediately.

We will apply the Schwarz symmetrization $\mathcal{S}_\xi K$ to $K$. Abusing notation, we will denote the new body again by $K$. We will write

$$t_0 = \min\{t \in \mathbb{R} : A_{K,\xi}(t) = \max_{t \in \mathbb{R}} A_{K,\xi}(t)\},$$

so that $K \cap (\xi^\perp + t_0\xi)$ is a section of $K$ orthogonal to $\xi$ of maximal volume. Since $0 < |K \cap H_\alpha| < \left|K \cap \left(\xi^\perp + t_0\xi\right)\right|$ we can find a cone with base equal to $K \cap \left(\xi^\perp + t_0\xi\right)$ and section equal to $K \cap H_\alpha$. Such a cone is uniquely determined by these two sections. Denote this cone by $N_1$. Let $\gamma\xi$ be the vertex of $N_1$, for some number $\gamma$ (either positive or negative). Due to the convexity of $K$, $\gamma\xi$ lies outside of $K$. Define $N_2$ to be the cone with base equal to $K \cap H_\alpha$ and vertex $\gamma\xi$; see Figure 4. Finally, we will let $H_\alpha^*$ be the halfspace bounded by the hyperplane $H_\alpha$ that contains $N_2$. We will consider two cases: $H_\alpha^* = H_\alpha^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \geq \alpha h_K(-\xi)\}$ and

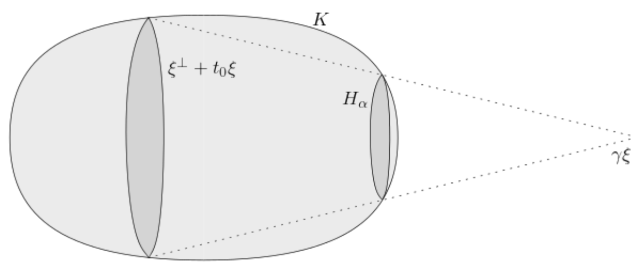


FIGURE 4. Constructing $N_1$ and $N_2$.

$H_\alpha^* = H_\alpha^- = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \leq \alpha h_K(-\xi)\}$. Denote $h = \alpha h_K(-\xi)$. When $H_\alpha^* = H_\alpha^+$ the following inequality holds:

$$|K \cap H_\alpha| = \frac{|N_2|\, n}{|\gamma - h|} \geq \frac{|K \cap H_\alpha^+|\, n}{|\gamma - h|}.$$

Then by Theorem 4 and using that $|K| = |K \cap H_\alpha^+| + |K \cap H_\alpha^-| \geq C_1(\alpha, n)\,|K| + |K \cap H_\alpha^-|$ we note that $\big(1 - C_1(\alpha, n)\big)|K| \geq |K \cap H_\alpha^-|$. We arrive at the following estimates

$$\begin{aligned}|K \cap H_\alpha| &\geq \frac{|K \cap H_\alpha^+|\,n}{|\gamma - h|} \geq C_1(\alpha, n)\frac{|K|\,n}{|\gamma - h|}\\ &\geq \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{|K \cap H_\alpha^-|\,n}{|\gamma - h|} \geq \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{|N_1 \setminus N_2|\,n}{|\gamma - h|}.\end{aligned}$$

Expressing the volumes of $N_1$ and $N_2$ in terms of their bases, we see

$$\begin{aligned}|K \cap H_\alpha| &\geq \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{(|N_1| - |N_2|)n}{|\gamma - h|}\\ &= \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\,|\gamma - t_0|}{|\gamma - h|} - \frac{C_1(\alpha, n)}{1 - C_1(\alpha, n)}\,|K \cap H_\alpha|\,.\end{aligned}$$

And so,

$$\begin{aligned}|K \cap H_\alpha| &\geq \frac{\frac{C_1(\alpha,n)}{1-C_1(\alpha,n)}}{1 + \frac{C_1(\alpha,n)}{1-C_1(\alpha,n)}}\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|}\\ &= C_1(\alpha, n)\frac{|K \cap (\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|}.\end{aligned}$$

Because $N_1$ is a homothetic copy of $N_2$, we can write

$$\frac{|\gamma - t_0|}{|\gamma - h|} = \frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|^{1/(n-1)}}{|K \cap H_\alpha|^{1/(n-1)}}.$$

Thus,

$$|K \cap H_\alpha| \geq C_1(\alpha, n)\,\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\,\frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|^{1/(n-1)}}{|K \cap H_\alpha|^{1/(n-1)}},$$

which implies

$$|K \cap H_\alpha| \geq C_1(\alpha, n)^{\frac{n-1}{n}}\,\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\,. \tag{8}$$

Now suppose $H_\alpha^* = H_\alpha^-$. Then the following inequality holds

$$|K \cap H_\alpha| = \frac{|N_2|\,n}{|\gamma - h|} \geq \frac{|K \cap H_\alpha^-|\,n}{|\gamma - h|}.$$

By Theorem 4 we have $\big((1 - C_2(\alpha, n)\big)|K| \leq |K \cap H_\alpha^-|$ and so the following inequalities hold

$$\begin{aligned}|K \cap H_\alpha| &\geq \frac{|K \cap H_\alpha^-|\,n}{|\gamma - h|} \geq \big(1 - C_2(\alpha, n)\big)\frac{|K|\,n}{|\gamma - h|}\\ &\geq \frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\frac{|K \cap H_\alpha^+|\,n}{|\gamma - h|} \geq \frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\frac{|N_1 \setminus N_2|\,n}{|\gamma - h|}.\end{aligned}$$

Expressing the volumes of $N_1$ and $N_2$ in terms of their bases, we get

$$\begin{aligned}|K \cap H_\alpha| &\geq \frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\frac{(|N_1| - |N_2|)n}{|\gamma - h|}\\ &= \frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\frac{\big|K \cap \big(\xi^\perp + t_0\xi\big)\big|\,|\gamma - t_0|}{|\gamma - h|} - \frac{1 - C_2(\alpha, n)}{C_2(\alpha, n)}\,|K \cap H_\alpha|\,.\end{aligned}$$

So,

$$|K\cap H_\alpha| \geq \frac{\frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}}{1+\frac{1-C_2(\alpha,n)}{C_2(\alpha,n)}} \frac{|K\cap(\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|}$$
$$= (1-C_2(\alpha,n))\,\frac{|K\cap(\xi^\perp + t_0\xi)||\gamma - t_0|}{|\gamma - h|}.$$

Again using the homothety of $N_1$ and $N_2$, we arrive at

$$|K\cap H_\alpha| \geq \big(1-C_2(\alpha,n)\big)\left|K\cap\big(\xi^\perp + t_0\xi\big)\right| \frac{\left|K\cap\big(\xi^\perp + t_0\xi\big)\right|^{1/(n-1)}}{|K\cap H_\alpha|^{1/(n-1)}},$$

which implies

$$|K\cap H_\alpha| \geq \big(1-C_2(\alpha,n)\big)^{\frac{n-1}{n}}\left|K\cap\big(\xi^\perp + t_0\xi\big)\right|. \tag{9}$$

Now to determine $D(\alpha,n)$ we need to find the minimum of the two constants in equations (8) and (9) for fixed $\alpha$. Note that $n\alpha \leq -\alpha$ for $\alpha\in(-1,0]$. Then it follows that

$$\big(1-C_2(\alpha,n)\big)^{\frac{n-1}{n}} = \left(\frac{n(\alpha+1)}{n+1}\right)^{n-1} \leq \left(\frac{n-\alpha}{n+1}\right)^{n-1} = C_1(\alpha,n)^{\frac{n-1}{n}},$$

for all $\alpha\in(-1,0]$, and thus we have our desired constant.

We will now consider $\alpha\in(0,1/n]$. We claim that the extremizing bodies are the same extremizing bodies in Theorem 4. Our plan of attack to prove this claim is to show that when we construct the bodies from Theorem 4, we can only decrease $|K\cap H_\alpha|$ and only increase the volume of the maximal section of $K$. We may prove this for the Schwarz symmetrization $\mathcal{S}_\xi K$ of $K$, which after abuse of notation we will denote by $K$. We will also employ Remark 3 and prove the result for $\overline{K}$ and $\overline{H}_\alpha = \{x\in\mathbb{R}^n : \langle x,\xi\rangle = (\alpha+1)\langle g(\overline{K}),\xi\rangle\}$. Again after abuse of notation, we will write $K$ for $\overline{K}$ and $H_\alpha$ for $\overline{H}_\alpha$. We will write

$$t_0 = \min\{t\in\mathbb{R} : A_{K,\xi}(t) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)\},$$

so that $K\cap(\xi^\perp + t_0\xi)$ is a section of $K$ orthogonal to $\xi$ of maximal volume. We will split the analysis into two parts, first for when $(\alpha+1)\langle g(K),\xi\rangle < t_0 < h_K(\xi)$, and then for when $0 < t_0 < (\alpha+1)\langle g(K),\xi\rangle$. The case $t_0 = (\alpha+1)\langle g(K),\xi\rangle$ is trivial.

Suppose that $(\alpha+1)\langle g(K),\xi\rangle < t_0 < h_K(\xi)$. Then following the upper bound construction in Theorem 4, we can construct a convex body

$$L = \mathrm{conv}(r_1 B_2^{n-1}, r_2 B_2^{n-1} + \mu\xi),$$

for some $r_1\geq 0$ and $r_2\geq 0$ such that $r_1+r_2>0$, and $\mu$ such that $(\alpha+1)\langle g(K),\xi\rangle < \mu < h_K(\xi)$, whose sections orthogonal to $\xi$ are Euclidean balls. Write

$$r_{K,\xi}(t) = A_{K,\xi}^{1/(n-1)}(t) \quad\text{and}\quad r_{L,\xi}(t) = A_{L,\xi}^{1/(n-1)}(t).$$

Lemma 1 tells us that $r_{K,\xi}$ and $r_{L,\xi}$ are concave on their support, and $r_{L,\xi}$ is affine on its support. In fact we can write (up to a constant depending only on $n$):

$$r_{L,\xi}(t) = \frac{r_2 - r_1}{\mu}t + r_1.$$

Since we are assuming $(\alpha+1)\langle g(K),\xi\rangle < t_0 < h_K(\xi)$, it follows that $r_2 \geq r_1$, and thus $A_{L,\xi}$ attains its maximum at $\mu$. Since $r_2 \geq r_1$ we get

$$\frac{r_2 - r_1}{\mu} \geq 0,$$

and so this, together with the fact from Theorem 4 that $\langle g(L),\xi\rangle \leq \langle g(K),\xi\rangle$, implies

$$|L \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle = (\alpha+1)\langle g(L),\xi\rangle\}| \\ \leq |K \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle = (\alpha+1)\langle g(K),\xi\rangle\}|\,.$$

We now want to show that the volume of the maximal section of $L$ is no smaller than the volume of the maximal section of $K$. Suppose the opposite, that is

$$A_{K,\xi}(t_0) > A_{L,\xi}(\mu).$$

Then it follows by the construction of $L$ and concavity of $r_{K,\xi}$ on its support that $\mu \leq t_0$. Raising both sides to the power $1/(n-1)$, we see

$$r_{K,\xi}(t_0) > r_{L,\xi}(\mu) = r_2.$$

Denote $f_K = (\alpha+1)\langle g(K),\xi\rangle$. Recall by the construction of $L$ we have $r_{K,\xi}(f_K) = r_{L,\xi}(f_K)$, and thus we can compute

$$\begin{aligned} |L \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \geq f_K\}| &= \int_{f_K}^{\mu} \left(\frac{r_2 - r_1}{\mu} t + r_1\right)^{n-1} dt \\ &= \int_{f_K}^{\mu} \left(\frac{r_2 - r_{K,\xi}(f_K)}{\mu - f_K}(t - f_K) + r_{K,\xi}(f_K)\right)^{n-1} dt = \frac{r_2^n - r_{K,\xi}^n(f_K)}{r_2 - r_{K,\xi}(f_K)} \frac{\mu - f_K}{n} \\ &\leq \frac{r_2^n - r_{K,\xi}^n(f_K)}{r_2 - r_{K,\xi}(f_K)} \frac{t_0 - f_K}{n} = \int_{f_K}^{t_0} \left(\frac{r_2 - r_{K,\xi}(f_K)}{t_0 - f_K}(t - f_K) + r_{K,\xi}(f_K)\right)^{n-1} dt, \end{aligned}$$

where we used $\mu \leq t_0$ for the above inequality. Denote

$$\zeta(t) = \left(\frac{r_2 - r_{K,\xi}(f_K)}{t_0 - f_K}(t - f_K) + r_{K,\xi}(f_K)\right).$$

Note that $\zeta(f_K) = r_{K,\xi}(f_K)$. Since by assumption $r_2 < r_{K,\xi}(t_0)$, it follows from concavity that $\zeta(t) < r_{K,\xi}(t)$ for all $t \in (f_K, t_0]$, and thus we have

$$\int_{f_K}^{t_0} \zeta^{n-1}(t)\,dt < \int_{f_K}^{t_0} A_{K,\xi}(t)\,dt = |K \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \geq f_K\}|\,.$$

Combining all of the above inequalities, we obtain

$$|L \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \geq f_K\}| < |K \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \geq f_K\}|\,,$$

a contradiction to our construction in Theorem 4. Therefore, we must have

$$A_{K,\xi}(t_0) \leq A_{L,\xi}(\mu),$$

as desired.

Without loss of generality, we may assume that $\mu = 1$. Then for $0 \leq t \leq 1$ we can write

$$A_{L,\xi}(t) = \big((r_2 - r_1)t + r_1\big)^{n-1} \tag{10}$$

where $r_2 > 0$ and $r_1 \geq 0$. Moreover, we may assume $r_2 > r_1$, otherwise there is nothing to prove. Then by Fubini's theorem and (10) we obtain

$$|L| = \int_0^1 A_{L,\xi}(t)\,dt = \frac{r_2^n - r_1^n}{(r_2 - r_1)n}.$$

and similarly we find

$$\langle g(L), \xi\rangle = \frac{1}{|L|}\int_0^1 tA_{L,\xi}(t)\,dt = \frac{r_1^{n+1} + \big((r_2 - r_1)n - r_1\big)r_2^n}{(r_2 - r_1)(n+1)(r_2^n - r_1^n)}.$$

Denote $f_L = (\alpha + 1)\langle g(L), \xi\rangle$. We are interested in minimizing

$$\varphi(r_1, r_2) = \left(\frac{(r_2 - r_1)f_L + r_1}{r_2}\right)^{n-1} = \frac{A_{L,\xi}(f_L)}{A_{L,\xi}(1)}.$$

As in Theorem 4, we may write $z = r_1/(r_2 - r_1)$ so that $\varphi$ can be written as a function of $z$:

$$\varphi(z) = \left(\frac{z + f_L}{z+1}\right)^{n-1},$$

where

$$f_L = (\alpha+1)\frac{z^{n+1} + (n - z)(z+1)^n}{(n+1)\big((z+1)^n - z^n\big)},$$

for $z \in [0, \infty)$. As in Theorem 4, computing the minimum of $\varphi$ is difficult, but we will work around this fact. For now, it is enough to note that $A_{L,\xi}(t)$ is increasing in $t$ on its support, so it follows that $A_{L,\xi}(\langle g(L), \xi\rangle) \leq A_{L,\xi}(f_L)$. And hence we obtain for $\alpha \in (0, 1/n]$ the following inequalities

$$\left(\frac{n}{n+1}\right)^{n-1} \leq \frac{A_{L,\xi}(\langle g(L), \xi\rangle)}{A_{L,\xi}(1)} \leq \min_{z \in [0,\infty)} \varphi(z), \tag{11}$$

where we used the result of Makai and Martini [7].

Now suppose that $0 < t_0 < (\alpha + 1)\langle g(K), \xi\rangle$. Then following the lower bound construction in Theorem 4, we can construct a convex body $M = \mathrm{conv}(0, rB_2^{n-1} + \beta\xi, v\xi)$ for some $v \geq h_K(\xi)$, $r > 0$, and $\beta$ such that $0 < \beta < (\alpha + 1)\langle g(K), \xi\rangle$, whose sections orthogonal to $\xi$ are Euclidean balls. Note that $M \cap (\xi^\perp + \beta\xi)$ is the maximal section of $M$ in the direction $\xi$. Similarly to above, we may write (up to a constant depending only on n):

$$r_{M,\xi}(t) = A_{M,\xi}^{1/(n-1)}(t) = \begin{cases} (r/\beta)t & t \in [0, \beta], \\ \frac{r}{\beta - v}(t - \beta) + r & t \in (\beta, v]. \end{cases}$$

Recall from Theorem 4 we proved $\langle g(M), \xi\rangle \geq \langle g(K), \xi\rangle$, and hence since $\beta - v < 0$, it follows that

$$\begin{aligned} &|M \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = (\alpha+1)\langle g(M), \xi\rangle\}| \\ &\qquad\qquad \leq |K \cap \{x \in \mathbb{R}^n : \langle x, \xi\rangle = (\alpha+1)\langle g(K), \xi\rangle\}|. \end{aligned}$$

Now we want to show that volume of the maximal section of $M$ is no smaller than the volume of the maximal section of $K$. Again, suppose the opposite, that is

$$A_{K,\xi}(t_0) > A_{M,\xi}(\beta).$$

Then it follows by the construction of $M$ and concavity of $r_{M,\xi}$ on its support that $\beta \geq t_0$. Raising both sides to the power $1/(n-1)$, we again obtain

$$r_{K,\xi}(t_0) > r_{M,\xi}(\beta) = r.$$

Denote $f_K = (\alpha+1)\langle g(K),\xi\rangle$. By the construction of $M$, we have $r_{K,\xi}(f_K) = r_{M,\xi}(f_K)$. Hence, we can compute

$$\begin{aligned}|M \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \leq f_K\}| &= \int_0^\beta \left(\frac{r}{\beta}\right)^{n-1} t^{n-1}\,dt \\ &\quad + \int_\beta^{f_K} \left(\frac{r}{\beta - v}(t-\beta) + r\right)^{n-1} dt \\ &= \int_0^\beta \left(\frac{r}{\beta}\right)^{n-1} t^{n-1}\,dt + \int_\beta^{f_K} \left(\frac{r - r_{K,\xi}(f_K)}{\beta - f_K}(t-\beta) + r\right)^{n-1} dt \\ &= \frac{\beta r^{n-1}}{n} + \frac{r_{K,\xi}^n(f_K) - r^n}{r_{K,\xi}(f_K) - r}\frac{f_K - \beta}{n} \\ &\leq \frac{t_0 r^{n-1}}{n} + \frac{r_{K,\xi}^n(f_K) - r^n}{r_{K,\xi}(f_K) - r}\frac{f_K - t_0}{n} \\ &= \int_0^{t_0} \left(\frac{r}{t_0}\right)^{n-1} t^{n-1}\,dt + \int_{t_0}^{f_K} \left(\frac{r - r_{K,\xi}(f_K)}{t_0 - f_K}(t-t_0) + r\right)^{n-1} dt,\end{aligned}$$

where we used $\beta \geq t_0$ for the above inequality. Write

$$\zeta(t) = \begin{cases} (r/t_0)t & t \in [0,t_0], \\ \frac{r - r_{K,\xi}(f_K)}{t_0 - f_K}(t-t_0) + r & t \in (t_0, f_K]. \end{cases}$$

Note that $\zeta(f_K) = r_{K,\xi}(f_K)$. Since by assumption $r < r_{K,\xi}(t_0)$, it follows from concavity that $\zeta(t) < r_{K,\xi}(t)$ for all $t \in [0, f_K)$, and thus we have

$$\int_0^{f_K} \zeta^{n-1}(t)\,dt < \int_0^{f_K} A_{K,\xi}(t)\,dt = |K \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \leq f_K\}|.$$

Combining all of the above inequalities, we obtain

$$|L \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \leq f_K\}| < |K \cap \{x \in \mathbb{R}^n : \langle x,\xi\rangle \leq f_K\}|,$$

a contradiction to our construction in Theorem 4. Therefore, we must have

$$A_{K,\xi}(t_0) \leq A_{L,\xi}(\beta),$$

and hence, it suffices to work with $M$ instead of $K$. Up to rescaling, we may assume that $h_M(\xi) = 1$ and $\left|rB_2^{n-1}\right| = n$. As before, we will define

$$M_2 = M \cap \{x \in \mathbb{R}^n \mid \langle x,\xi\rangle \geq \beta\}.$$

Taking our computations from Theorem 4, we have $|M_2| = 1 - \beta$ and

$$\left|M \cap H_\alpha^+\right| = \frac{(1-f_M)^n}{(1-\beta)^{n-1}},$$

where

$$f_M = (\alpha+1)\langle g(M),\xi\rangle = \frac{\beta(n-1)+1}{n+1}.$$

Then by expressing the volumes of the sections we are interested in with respect to $M_2$ and $M \cap H_\alpha^+$, we can write

$$\frac{|M \cap \{x \in \mathbb{R}^n : \langle x, \xi \rangle = f_M\}|}{\big|M \cap (\xi^\perp + \beta\xi)\big|} = \frac{|M \cap H_\alpha^+|}{|M_2|} \frac{1-\beta}{1-f_M}$$
$$= \left(\frac{1-f_M}{1-\beta}\right)^{n-1}.$$

Denote by $\psi$ the above equation of $\beta$. Our goal is to find the infimum of $\psi$ on $(0, \frac{\alpha+1}{-(n-1)\alpha+2})$ (as taken from Theorem 4). One can compute that $\psi$ is increasing in $\beta$ and therefore the infimum of $\psi$ is given by

$$\psi(\beta) \xrightarrow{\beta \to 0^+} \left(\frac{n-\alpha}{n+1}\right)^{n-1}. \tag{12}$$

Now to determine the value of $D(\alpha, n)$ for $\alpha \in (0, 1/n]$, we need to find the lower of the two constants in (11) and (12). It is enough to note for $\alpha \in (0, 1/n]$ that

$$\left(\frac{n-\alpha}{n+1}\right)^{n-1} \le \left(\frac{n}{n+1}\right)^{n-1},$$

and the result follows.

Discussing equality cases, we see for $\alpha \in (-1, 0]$ that equality follows from the equality cases for the upper bound in Theorem 3 (which comes from Grünbaum's original theorem), and thus the equality bodies are, up to translation, cones of the form $L = \text{conv}(B, v)$ with $B$ an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in $\xi^+$ and vertex $v$ lying in $\xi^-$. For $\alpha \in (0, 1/n]$, we have our convex body $M$ which is the union of two cones with a common base at $\beta\xi$. We have equality in the limit when $\beta \longrightarrow 0^+$, thus the equality bodies are cones of the form $M = \text{conv}(B, v)$ with $B$ an $(n-1)$-dimensional convex body lying parallel to $\xi^\perp$ in $\xi^-$ and vertex $v$ lying in $\xi^+$; see Figure 5. □

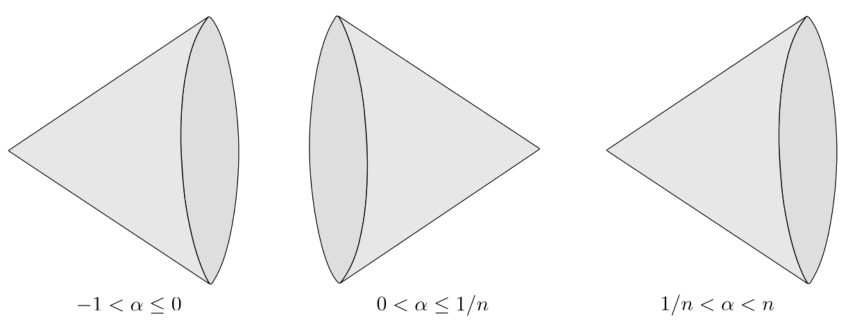


FIGURE 5. Equality cases for Theorem 5.

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