Topology Notes

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0.1 Topology

0.1.1 Introduction

A *topological space* is a set X equipped with a *topology* $\tau \subset 2^X$ such that $\varnothing, X \in \tau$, and τ is closed under arbitrary unions and finite intersections. For any set X we can consider the *trivial topology* $\tau = \{\varnothing, X\}$ and the *discrete topology* $\tau = 2^X$. Elements of τ are called *open sets*, while elements whose complement is in τ are called *closed sets*. Given two topologies τ_1, τ_2 on a set X we say that τ_1 is *weaker* than τ_2 if $\tau_1 \subset \tau_2$. τ_2 is said to be *stronger* than τ_1. As a remark, it is easy to find examples of topologies which are not comparable.

If X is a metric space, then X induces a topology τ where a set $U \subset X$ is open if for all $x \in X$, there is an open ball $B_\varepsilon(x)$ for some $\varepsilon > 0$ such that $x \in B_\varepsilon(x) \subset U$. The set of open balls in X has the property that for any two two non-disjoint open balls $B_\varepsilon(x), B_\delta(y)$, for $\varepsilon, \delta > 0$ and $x, y \in X$, for all $z \in B_\varepsilon(x) \cap B_\delta(y)$ there is an open ball $B_\theta(z)$ for $\theta > 0$ such that $z \in B_\theta(z) \subset B_\varepsilon(x) \cap B_\delta(y)$. We call this the *metric topology* of X.

With this in mind, we say that a set $B \subset 2^X$ is a *basis* if B covers X and for any two non-disjoint $A_1, A_2 \in B$ and $x \in A_1 \cap A_2$ there is $A_3 \in B$ such that $x \in A_3 \subset A_1 \cap A_2$. Every basis B gives rise to a topology τ where $U \subset X$ is open if for all $x \in U$ there is $A \in B$ such that $x \in A \subset U$. Every topology τ has a basis, namely itself and every basis B can be completed to a topology by closing B under arbitrary unions.

Lemma 1. *Let B be a basis for X. Then the topology τ induced by B is equal to the closure of B under arbitrary unions.*

Proof. If we have a collection of elements in B then they are open in τ and so their union is open. Coversely if we are given an open set U in τ then $U = \bigcup_{x \in U} V_x$ for elements $V_x \in B$ and so we see the reverse inclusion. $\square$

Interestingly as well, if X is a topological space and B is a collection of open sets such that for all open sets U and $x \in U$ there is $A \in B$ such that $x \in A \subset U$ then B is a basis. This is not too hard to prove.

Lemma 2. *Let τ_1, τ_2 be two topologies on X with corresponding basis B_1, B_2. The following are equivalent:*

1. *τ_1 is weaker than τ_2*
2. *for each $A_1 \in B_1$ and $x \in A_1$ there is $A_2 \in B_2$ such that $x \in A_2 \subset A_1$.*

Proof. If τ_1 is weaker than τ_2 then let $A_1 \in B_1$ and $x \in A_1$. Then $A_1 \in \tau_1 \subset \tau_2$ so there exists $A_2 \in \tau_2$ such that $x \in A_2 \subset A_1$.

On the other hand, let $U \in \tau_1$. Then for each $x \in U$ there is $A_1 \in B_1$ such that $x \in A_1 \subset U$. Then by assumption there is $A_2 \in B_2$ such that $x \in A_2 \subset U$. Since x is arbitrary, $U \in \tau_2$. $\square$

A *sub-basis* B for a topology is a cover for X. Every sub-base induces a basis by closing B under finite intersections, and thus in term yields a topology in the sense for a basis. Now that we understand this, we can make sense of two topologies which are not comparible.

Lemma 3. *Let $\mathbb{R}, \mathbb{R}_l, \mathbb{R}_u$ be the set of real numbers equipped with the standard, lower, and upper topologies. Then $\mathbb{R}_l$ and $\mathbb{R}_u$ are not comparible but both are strictly stronger than $\mathbb{R}$.*

Proof. Easy. $\square$

0.1.2 Subspace, Product, Order Topologies

Given an arbitrary set of topological spaces $\{X_\alpha\}$ we define the *product topology* on

$$\prod X_\alpha$$

where open sets are of the form

$$\prod U_\alpha$$

where $U_\alpha = X_\alpha$ except for finitely many α. One can check that

$$B = \{\prod U_\alpha : U_\alpha = X_\alpha \text{ except for finitely many } \alpha\}$$

is a basis for $\prod X_\alpha$. We define the projection map canonically by

$$\pi_\beta : \prod X_\alpha \longrightarrow X_\beta.$$

One sees that $\pi_\beta^{-1}(U)$. The set of all inverse images of U_β under π_β as we run through β forms a sub-base of the product topology. If we consider the *box topology* on the product space (i.e. where the topology is formed by products of all open sets then clearly the product topology is weaker than the box topology, but also the box topology is strictly stronger in some cases since in $\mathbb{R}^{\mathbb{N}}$ we have the open set

$$0 \in (1,1) \times (1,1) \times (1,1) \times \ldots$$

and there is no neighbourhood of the product topology contained in this open set.

The *subspace topology* is of a topological space X is defined using a set A by considering the restriction of the topology of sets to A. For example, $[0,1]$ inherits a subspace topology from the natural topology on $\mathbb{R}$ where $(0,1]$ is open. A set $U \cap A$ is open in the subspace topology if U is open in the regular topology. On the converse, the singleton $\{1\}$ is closed in $\mathbb{R}$ and open in its subspace topology. Given a basis, resp. subbasis, we can intersect with A to obtain a basis for the subspace topology.

The *order topology* of a totally ordered set X is defined by considering a sub-basis of rays of the form $\{x \in X : x > \alpha\}$ or $\{x \in X : x < \alpha\}$ for $\alpha \in X$.

0.1.3 Continuity

We say that a map of topological spaces $f : X \longrightarrow Y$ is continuous if the preimage of an open set in Y is open in X. For example, any map from a discrete topological space to another is continuous. If $\tau_1 \leq \tau_2$ are two topologies then continuity with τ_2 implies continuity with τ_1 (in the domain, which means in other words that continuity goes down).

The product topology is the weakest topology for which all projection maps are continuous. If all projection maps are continuous then we clearly get openness in the product topology and if we had another topology with this property then for any open set $U_\alpha \subset X_\alpha$ one has that $\pi_\alpha^{-1}(U_\alpha)$ is open. Consequently members $\pi_\alpha^{-1}(U_\alpha)$ form a sub-basis but since this a sub-base of the product topology they must agree.

0.1.4 Closed Sets

Lemma 4. *A set A is closed in the subspace topology of Y means that $A = Y \cap V$ for some X closed V.*

Proof. A is closed so $Y \setminus A$ is open and thus $Y \setminus A = Y \cap U$ for some U open. Then $A = Y \setminus (Y \setminus A) = Y \setminus (Y \cap U) = Y \setminus U = Y \cap (X \setminus U)$. Then $X \setminus U = V$ is the closed set we want. $\square$

The *closure* of A in X is the smallest closed set in X which contains A. Alternatively it is the intersection of all closed spaces in X containing A. The *interior* of A is the largest open set contained in A. We denote the closure by $\overline{A}$ and interior by A°. Alternatively we define the closure as the set of all closure points of A, i.e. elements $x \in X$ for which all neighbourhoods of x meet A. A is closed iff A is equal to its closure.

0.1.5 Hausdorff Axiom

We say a topological space is *Hausdorff* if any two distinct points of X can be seperated by neighbourhoods. Being a Hasudorff space means that all singletons are closed. Indeed, given $x \in X$ if $\{x\}$ is not closed then let $y \in \overline{\{x\}}$. Then by the Hausdorff axiom if $x \neq y$ we can seperate x by a neighbourhood of y, contradicting the fact that its in the closure.

We say that a net x_α converges to some $x \in X$ if every neighbourhood of x contains a tail of x_α. We can think of the same with sequences. Given a function f, we can say that f is continuous if it maps convergent nets to convergent nets, or alternatively if the preimage of closed sets are closed, or if $f(\overline{A}) \subset \overline{f}(A)$ for all A, or if for all $x \in X$ and $V \ni f(x)$ there is $U \ni x$. such that $f(U) \subset V$.

A homoemorphism between two topological spaces X and Y is a map $f : X \longrightarrow Y$ which is continuous, bijective, and has continuous inverse.

Lemma 5. *If $f : A \longrightarrow Y$ and $g : B \longrightarrow Y$ are two continuous functions which agree on $A \cap B$ then $f \cup g$ is continuous.*

As a corollary, f is continuous iff $\pi_\alpha \circ f$ is continuous for all α in the product topology.

0.1.6 Metrizability

The box topology is not metrizable. If it were metrizable then let $A = \{x : x_i > 0\}$. Then $0 \in \overline{A}$ but for any sequence in A we can find a neighbourhood of 0 that contains no tail of the sequence.

$\mathbb{R}^{\mathbb{N}}$ is metrizable under the product topology with metric

$$D(x, y) = \sup_i d(\frac{\min(x_i - y_i, 1)}{)}.$$

This gives rise to a topology on $\mathbb{R}^{\mathbb{N}}$ called the uniform topology. One has

$$\mathbb{R}^{\mathbb{N}}_{prod} \subsetneq \mathbb{R}^{\mathbb{N}}_{unif} \subsetneq \mathbb{R}^{\mathbb{N}}_{box}.$$

When one deals with metrizability, closures can be discussed in terms of limits of sequences, but when there is no metrizability, one has to use nets.

0.1.7 Quotient Maps

A *quotient map* is a map $\rho : X \longrightarrow Y$ which is surjective and continuous and $V \subset Y$ is open iff for every $\rho^{-1}(V)$ is open.

Lemma 6. *If ρ is a surjection from a topological space X to a set Y there exists a unqiue topology on Y such that ρ is a quotient map.*

Proof. Define $\tau_Y = \{V \subset Y : \rho^{-1}(V) \text{ is open}\}$. Topology on Y. □

Using this, we can say that if $\rho : X \longrightarrow Y$ is a quotient map and $g : X \longrightarrow Z$ is a map. There is a map $f : Y \longrightarrow Z$ (of sets) such that $f \circ \rho = g$. f is continuous iff g is and a quotient iff g is. As ana pplication if we have maps $Z \longrightarrow X$ and maps $Z \longrightarrow Y$ then there is a map $Z \longrightarrow X \times Y$ whose projection agrees with the original map. Then given a topological space X and an equivalence relation $\sim$ on X we can define the quotient space of X by $X/\sim$ equipped with the topology where $U \subset X/\sim$ is open iff $\rho^{-1}(U)$ is open in X.

0.1.8 Connectedness

A topological space is connected if it isn't the union of two disjoint non-empty open sets. X is connected iff the only open and closed sets of X are trivial. If X is connected and there is such a set which is non trivial then $X = U \cup U^c$ is a disjoint union of two non-empty open sets. Moreover, if X is not connected then X is the disjoint union of $U \cup V$ where $U \cup V$ are open disjoint and non-empty. Then $X \setminus U = V$ is closed, and non-empty, but also open.

If f is continuous and U is connected, then $f(U)$ is connected. Moreover, homeomorphisms preserve connectedness. If X is disconnected, then any set $A \subset X$ which is connected must lie in a *connected component*, i.e. a maximal connected set which contains A or alternatively the connected components are equivalence classes modulo the relation $x \sim y$ if there exists a connected subset of X which contains x and y. If $\{A_\alpha\}$ are a collection of connected sets whose intersection is non-empty then their union is connected. In this sense we can see that $X \times Y$ is connected provided both X and Y are. One can see that $R^{\mathbb{N}}$ is not connected in the box topology because the set of all bounded sequences is open and the set of all unbounded sequences is open.

A topological space is path connected if for all $x, y \in X$ there is a continuous path $[0,1] \longrightarrow X$ between x and y. The topologists sine curve is an example of a connected set which is not path connected. The closure of a connected set is always connected.

0.1.9 Compactness

A space X is compact if every open cover has a finite subcover. Alternatively every net has a convergent subnet. Closed subspaces of compact spaces are compact.

Theorem 7. *Suppose that X is Hausdorff and that $Y \subset X$ is compact. Then Y is closed.*

Proof. For every $y \notin Y$ find neighbourhoods which seperate Y from y. By compactness Y has a finite subcover and take interseections over y to find two neighbourhoods which seperate y and Y. $\square$

Compactness is preserved under homeomorphisms. We have Heine-Borel, and Extreme Value Theorem. Also any continuous function on a compact domain is uniformly continuous (between metric spaces).

Theorem 8. *The product of an arbitrary amount of compact sets is compact in the product topology.*

X is compact iff for every closed collection $\{A_\alpha\}$ satisfying the finite intersection property we have that $\bigcap A_\alpha \neq \varnothing$. (The finite intersection property is that all finite subsets of $\{A_\alpha\}$ don't intersect emptily.

0.1.10 Countability

A topological space X is first countable if every $x \in X$ has a countable base of neighbourhoods (every neighbourhood contains one of these neighbourhoods). A topological space is second countable if there is a countable basis. Second countable implies first countable.

Why is this nice? Well if we have first countable then we can work with sequences. $\mathbb{R}^{\mathbb{N}}$ with uniform topology is first countable (since it is metrizable) but not second countable under the uniform topology since every discrete set must be countable but the set of all sequences of zeros and ones is uncountable but discrete. If X is second countable then every open cover has a countable subcover.

Normal seperates closed sets. Regular seperates points and closed sets. Hausdorff seperates points. (In normal and regular defn we assume that points are closed). Every compact Hausdorff space is normal. Every metrizable space is normal.

Lemma 9. *X is regular iff for every $x \in X$ and each $U \ni x$ there is $V \ni x$ such that $\overline{V} \subset U$. X is normal iff for every $A \subset X$ closed and $A \subset U$ there is $A \subset V$ such that $\overline{V} \subset U$.*

Lemma 10 (Urysohn's Lemma). *If X is normal and $A, B \subset X$ are disjoint and closed then there is $f : X \longrightarrow [0, 1]$ continuous such that $f(A) = 0$, $f(B) = 1$.*

this is just seperation.

Theorem 11 (Urysohns). *Every second countable regular space is metrizable.*

0.1.11 Baire Spaces

A space X is said to be Baire if every countable intersection of open dense set is again dense. Alternatively every union of closed sets with empty interior has empty interior.

Theorem 12. *Every complete metric space is Baire. Every compact Hausdorff space is Baire.*

Theorem 13. *Continuous and non-differentiable functions are dense in $C[0, 1]$.*

Proof. Let $h > 0$ and $\Delta f(x, h) = \max\{\frac{f(x+h)-f(x)}{h}, \frac{f(x-h)-f(x)}{-h}\}$ and $\Delta f = \inf_{x\in[0,1]} \Delta f(x, h)$. Δf is continuous in h. Let $U_n = \{f \in C[0, 1] : \Delta f_n > n$ for some $h \leq 1/n\}$. Then U_n is open and desne. is open and dense. Now if $f \in \bigcap_{n=1}^{\infty} U_n$ then for any $t \in [0, 1]$ and n there existst $0 < |h_n| < 1$ such $|\frac{f(t+h_n)-f(t)}{h_n}| > n$ so we are done. $\square$

0.2 Algebraic Topology

0.2.1 Fundemental Groups

In this section we let X and Y be topological spaces and $I = [0, 1]$. Denote by $C(X, Y)$ the set of all continuous maps $f : X \longrightarrow Y$.

Definition 1. Let $f, g \in C(X, Y)$. We say f is *homotopic* to g (written $f \sim g$) if there exists a continuous map $F : X \times I \longrightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

In the above definition, F is called a *homotopy* between f and g. If $f \in C(X, Y)$ and f is homotopic to a constant map, then we say that f is *nulhomotopic*.

Definition 2. Let $x_1, x_2 \in X$ and $\gamma_1, \gamma_2 \in C(I, X)$ be continuous curves between x_1 and x_2. We say that γ_1 is *path-homotopic* to γ_2 (written $\gamma_1 \sim_p \gamma_2$) if there exists a continuous map $F : I^2 \longrightarrow X$ such that $F(0, t) = x_1$, $F(1, t) = x_2$, $F(x, 0) = \gamma_1(x)$ and $F(x, 1) = \gamma_2(x)$.

Using this notation, we say that x_1 is the *initial point* and x_2 is the *final point*.

Lemma 14. *$\sim$ and $\sim_p$ are equivalence relations on continuous maps and curves.*

Proof. Easy. Just draw a picture and figure out the exact mappings that give you reflexivity, symmetry, and transitivity of $\sim$ (resp. $\sim_p$). □

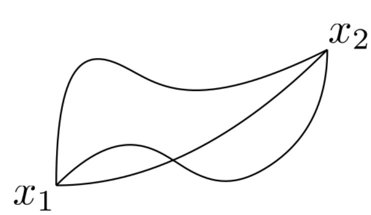

Figure 1: Three path-homotopic curves between $x_1, x_2 \in X$.

When Y is convex then all maps $X \longrightarrow Y$ are homotopic via. the canonical straight-line homotopy.

Definition 3. Let $x_1, x_2, x_3 \in X$ and $\gamma_1 \in C(I, X)$ be a continuous curve from x_1 to x_2, while $\gamma_2 \in C(I, X)$ is a continuous curve from x_2 to x_3. We define by $\gamma_1 \cdot \gamma_2$ the product of γ_1 and γ_2, a curve from x_1 to x_3 defined by

$$(\gamma_1 \cdot \gamma_2)(t) = \begin{cases} \gamma_1(2t) : t \in [0, 1/2], \\ \gamma_2(2t - 1) : t \in [1/2, 1], \end{cases}$$

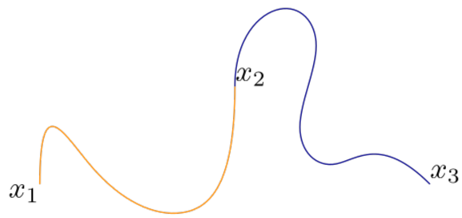

Figure 2: The product of two curves.

Note that this product is well defined at $t = 1/2$ by the gluing lemma. See below for an illustration of such a product. Given the product operation on continuous curves, we can induce a natural product operation on equivalence classes of curves modulo homotopy. Indeed, we can define $[f] \cdot [g] = [f \cdot g]$ for $[f], [g] \in C(I, X)/\sim_p$. Such an operation is well defined because if $f \sim_p f'$ and $g \sim_p g'$ then $f \cdot g \sim f' \cdot g'$. This is just an application of once again drawing a picture out and figuring out the proper homotopy between $f \cdot g$ and $f' \cdot g'$. See the below illustration. Note that the product has the following properties.

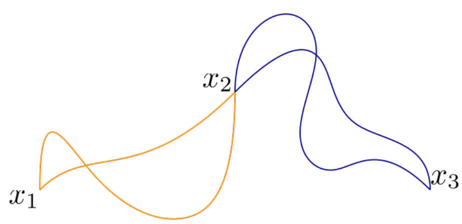

Figure 3: The product on equivalence classes of continuous curves modulo homotopy is well-defined.

First, the product is associative. Furthermore, we have a left and right identity. Finally, we have the inverses. We will dive into this below.

Lemma 15. *The product is associative. Moreover there exists left and right identities and inverses.*

Proof. We remark that the product being associative means that

$$(f \cdot g) \cdot h \sim_p f \cdot (g \cdot h),$$

for $f, g, h \in C(I, X)$ being continuous curves from x_1 to x_2 and then x_2 to x_3 and then x_3 to x_4. We can write

$$((f \cdot g) \cdot h)(t) = \begin{cases} f(4t) : t \in [0, 1/4], \\ g(4t-1) : t \in [1/4, 1/2], \\ h(2t-1) : t \in [1/2, 1]. \end{cases}$$

Further we write

$$(f \cdot (g \cdot h))(t) = \begin{cases} f(2t) : t \in [0, 1/2], \\ g(4t-2) : t \in [1/2, 3/4], \\ h(4t-3) : t \in [3/4, 1]. \end{cases}$$

One can verify easily that these maps are homotopic. □

The left identity is given by $[e_{x_1}]$ where $e_{x_1} : I \longrightarrow X$ is the constant curve $e_{x_1}(t) = x_1$. The right identity is given similarly. The inverse of $[f]$ is given by $[\overline{f}]$ where $\overline{f}(t) = f(1-t)$.

Definition 4. Let $x_1 \in X$. A continuous curve that begins and ends at x_1 is called a loop at x_1. The set of all loops at x_1 modulo homotopy (denoted by $\pi_1(X, x_1)$) equipped with the product above is called the *fundemental group* at x_1.

Indeed, this set becomes a group under the above product. Every left/right identity simply becomes an identity when considering loops at x_1. Consider the following illustration below. Let $\gamma : [0, 1] \longrightarrow X$ be a path between $x_1, x_2 \in X$.

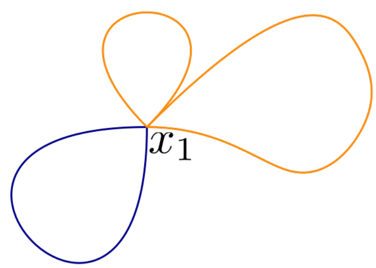

Figure 4: The fundemental group at x_1.

Then γ induces a map $\gamma^* : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_2)$ given by $\gamma^*([f]) = [\overline{\gamma}][f][\gamma]$. Again it is easy to see that such a map is well defined because the product is well defined. One verifies that this map defines a group isomorphism, and thus this shows a further corollary:

Corollary 1. *Suppose that X is path-connected. Then $\pi_1(X, x_1) \cong \pi_1(X, x_2)$ for all $x_1, x_2 \in X$.*

Note that just because all of these groups are isomorphic, the isomorphism isn't necessarily independent of the underlying continuous path. It is not too hard to see that the isomorphism is independent of the path iff the fundemental group is abelian.

Definition 5. If X is path-connected, we say that X is *simply-connected* if the fundemental groups of X are trivial.

One can think of a simply-connected topological space as a path-connected space for which every loop can be continuously deformed to a single point.

Lemma 16. *In a simply connected space X where $x_1, x_2 \in X$ one has that any two continuous paths from x_1 to x_2 are homotopic.*

Proof. Let γ_1, γ_2 be two paths from x_1 to x_1. Then $\gamma_1 \cdot \overline{\gamma}_2$ is a loop at x_1 and thus $[\gamma_1 \cdot \overline{\gamma}_2] = [e_{x_1}]$. Multiply both sides by $[\gamma_2]$ to obtain $[\gamma_1] = [\gamma_2]$. □

Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. If we fix $x_1 \in X$ and consider $y_1 = f(x_1)$ then we have an induced map

$$f_* : \pi_1(X, x_1) \longrightarrow \pi_1(Y, y_1)$$

defined by $f_*([\gamma]) = [f \circ \gamma]$. This map is well defined for if γ_1, γ_2 are two loops at x_1 which are homotopic under a homotopy F then $f \circ \gamma_1$ and $f \circ \gamma_2$ are homotopic loops at y_1 with homotopy $f \circ F$. This induced map is called the *pushforward* of f at x_1.

Lemma 17. *Let $f \in C(X, Y)$ and $g \in C(Y, Z)$ be continuous maps from a topological space X to Y to Z. Then $(g \circ f)_* = g_* \circ f_*$ and further $\iota_* = \iota$ is the identity homomorphism where $\iota : X \longrightarrow X$ is the identity map.*

We have a simple corollary of the above Lemma:

Corollary 2. *If X is homeomorphic to Y then $\pi_1(X, x_1)$ is isomorphic to $\pi_1(Y, y_1)$ where $x_1 \in X$, $y_1 = f(x_1)$ and $f : X \longrightarrow Y$ is the homeomorphism between X and Y.*

Proof. Just verify that f_*^{-1} is the required inverse homomorphism to f_*. □

0.2.2 Covering Spaces

In this section, we will be wanting to study how exactly we can compute fundemental groups of certain topological spaces. For a continuous surjective map $\rho : E \longrightarrow X$ where E and X are topological spaces, we say that a neighbourhood U of X is *evenly covered* in E if $\rho^{-1}(U)$ can be expressed as a disjoint union of open sets

$$\bigcup_{\alpha \in A} V_\alpha, \quad V_\alpha \subset E,$$

where ρ restricted to each V_α is a homeomorphism of U. This prompts the following definition

Definition 6. A *covering space* of a topological space X is a topological space E equipped with a continuous surjection $\rho : E \longrightarrow X$ such that every $x \in X$ has a neighbourhood which is evenly covered in E.

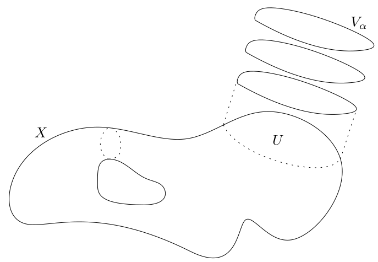

Figure 5: Covering spaces.

We call ρ a *covering map* of X from E. Geometrically, we may view a covering space in the illustration below. As an example, consider the covering space $E = \mathbb{R}$ for $X = S^1$. The covering map is given by $\rho : \mathbb{R} \longrightarrow S^1$ defined by $\rho(\theta) = (\cos 2\pi\theta, \sin 2\pi\theta)$. One can convince themselves that this is indeed a covering map. If we have a covering map $\rho : E \longrightarrow X$ then ρ is a local homeomorphism, that is, for all $e \in E$ there is a neighbourhood of e which is homeomorphic to a subset of X under ρ. The restriction of a covering map may not be a covering map, because for example

$$\rho : \mathbb{R} \longrightarrow S^1$$

is a covering map but its restriction to $\mathbb{R}_{\geq 0}$ is not. Fortunately we have the following Lemma:

Lemma 18. *Let $\rho : E \longrightarrow X$ be a covering map. Let Y be a subspace of X. Then $\overline{\rho} : \rho^{-1}(Y) \longrightarrow Y$ is a covering map.*

Proof. By definition, this is a surjective continuous map. Further, if $y \in Y$ then $y \in X$ and so there exists a neighbourhood $U \ni y$ of X such that $\rho^{-1}(U) = \bigcup_{\alpha \in A} V_\alpha$ for a disjoint union of open V_α. Then

$$\rho^{-1}(U \cap Y) = \bigcup_{\alpha \in A} V_\alpha \cap \rho^{-1}(Y),$$

where $y \in U \cap Y$ is a neighbourhood of y and $V_\alpha \cap \rho^{-1}(Y)$ is a disjoint union of open neighbourhoods in $p^{-1}(Y)$. The homeomorphism between $V_\alpha \cap \rho^{-1}(Y)$ and $U \cap Y$ is due to ρ. $\square$

Furthermore, if $\rho_1 : E_1 \longrightarrow X_1$ and $\rho_2 : E_2 \longrightarrow X_2$ are covering maps then so is

$$\rho = d(\rho_1, \rho_2) : E_1 \times E_2 \longrightarrow X_1 \times X_2.$$

As an example, if we consider the covering map

$$\mathbb{R} \times \mathbb{R}_{>0} \longrightarrow S^1 \times \mathbb{R}_{>0}$$

then we can compose this with the canonical homeomorphism from $S^1 \times \mathbb{R}_{>0}$ to $\mathbb{R}^2 \setminus \{0\}$ by sending $(x, r) \longrightarrow rx \in \mathbb{R}^2 \setminus \{0\}$. This gives a covering map

$$\mathbb{R} \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}^2 \setminus \{0\}.$$

See the illustration below for a geometrical argument as to what this covering space looks like. One can imagine in this image each point on the loop about

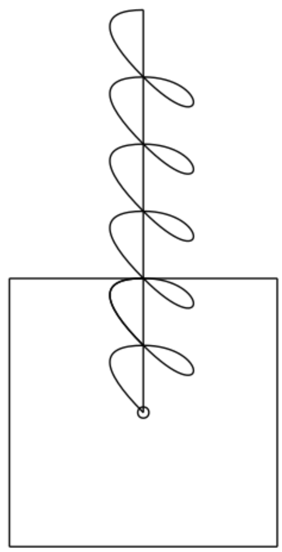

Figure 6: Covering $\mathbb{R}^2 \setminus \{0\}$ by $\mathbb{R} \times \mathbb{R}_{>0}$.

the origin is connected to the pole in the center by strips of $\mathbb{R}_{>0}$.

Definition 7. Let Y be a topological space and $\rho : E \longrightarrow Y$ be a continuous map of topological spaces. If $f : X \longrightarrow Y$ is a continuous map of topological spaces then we say that f *lifts* to E if there is a continuous map of topological spaces $\overline{f} : X \longrightarrow E$ such that $f = \rho \circ \overline{f}$.

Our goal is to show that in the case of covering spaces that paths can be lifted, and so can path homotopies. For an example, if we consider the curve $f : [0, 1] \longrightarrow S^1$ given by

$$f(\theta) = (\cos \pi\theta, \sin \pi\theta),$$

lifts to $\mathbb{R}$ via. $\overline{f} : [0, 1] \longrightarrow \mathbb{R}$ through $\overline{f}(t) = t/2$, where ρ is the canonical covering of $\mathbb{R}$ on S^1.

Theorem 19. *Let $\rho : E \longrightarrow X$ be a covering map. Let $e \in E$ and $x = \rho(e)$. Any path $f : [0, 1] \longrightarrow X$ beginning at x can be lifted uniquely to a path in E beginning at e.*

Proof. Let $\mathcal{U}$ be a covering of X by open sets for which every $U \in \mathcal{U}$ is evenly covered. Choose a subdivision of $[0,1]$, say $[0,s_1],[s_1,s_2],\dots,[s_{n-1},1]$ such that $f([s_k,s_{k+1}]) \subset U$ for some $U \in \mathcal{U}$. We define the lifting $\overline{f}$ step by step. First define $\overline{f}(0) = e$. If we suppose that $\overline{f}$ is defined on $[0,s_k]$ then we define $\overline{f}$ on $[s_k,s_{k+1}]$ as follows. Let $U \in \mathcal{U}$ be chosen so that $f([s_k,s_{k+1}]) \subset U$. Then $\rho^{-1}(U) = \bigcup_{\alpha\in A} V_\alpha$ for a disjoint union of open V_α. Then by disjointness $\overline{f}(s_k)$ lies in one of these sets, say V_0 and thus we define $\overline{f}(s)$ for $s \in [s_k,s_{k+1}]$ as follows:

$$\overline{f}(s) = \rho^{-1}_{|V_0}(f(s)).$$

$\overline{f}$ is continuous because $\rho^{-1}_{|V_0}$ is a homeomorphism between V_0 and U. The continuity of $\overline{f}$ follows by the pasting lemma and the fact that $f = \rho \circ \overline{f}$ is clear. The uniqueness follows by a similar argument. If we assume that $\overline{g}$ is another lifting of f then $\overline{g}(0) = \overline{f}(0)$ and thus if we assume that $\overline{f}(s_k) = \overline{g}(s_k)$ then we see since each V_α is disjoint and $[s_k,s_{k+1}]$ is connected that $\overline{g}([s_k,s_{k+1}]) \subset V_0$. The uniqueness routinely follows because $\rho^{-1}_{V_0}$ is a homeomorphism. □

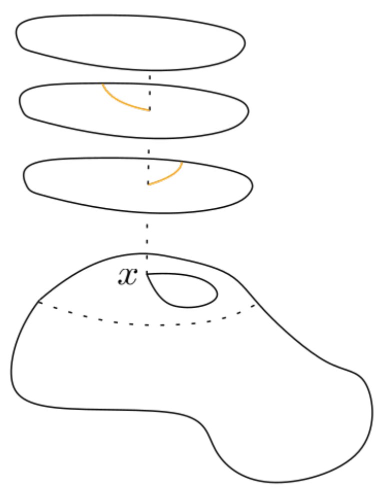

Figure 7: Path lifting.

Do note that although such a path can be lifted, the end point may not be what we want. For example a lifting of a loop may not be a loop. Fortunately we have the following result though

Lemma 20. *Let $F : I \times I \longrightarrow X$ be a path homotopy of two curves with $F(0,0) = x \in X$. Let $\rho : E \longrightarrow X$ be a covering map of X and $e \in E$ such that $\rho(e) = x$. Then F lifts to a path homotopy $\overline{F}$ such that $\overline{F}(0,0) = e$.*

Proof. □

Theorem 21. *Let $f, g : [0,1] \longrightarrow X$ be two continuous paths from x_1 to x_2 which are path homotopic where $x_1 = \rho(e)$ for some $e \in E$ and E is a covering space of X. Consider the liftings $\overline{f}$ and $\overline{g}$. Then these start at e and end at the same point, and further are path homotopic.*

Proof. □

This prompts the following definition. If $\rho : E \longrightarrow X$ is a covering map and $x \in X$ we choose $e \in E$ such that $\rho(e) = x$. Then the *lifting correspondence* $\varphi : \pi_1(X, x) \longrightarrow \rho^{-1}(x)$ defined by $\rho([f]) = \overline{f}(1)$ is a well defined map. If E is path-connected then φ is surjective. If E is simply-connected then φ is bijective.

Proof. If E is path connected Then given $e_1 \in \rho^{-1}(x)$ there is a path $\overline{f} : [0,1] \longrightarrow E$ between e and e_1. Then $f = \rho \circ \overline{f}$ is a loop in X at x and $\varphi([f]) = \overline{f}(1) = e_1$.

If E is simply connected then let $[f], [g] \in \pi_1(X, x)$ such that $\varphi([f]) = \varphi([g])$. Then this means that $\overline{f}(1) = \overline{g}(1)$. Since E is simply connected, there is a path-homotopy F between $\overline{f}$ and $\overline{g}$. Then $\rho \circ F$ is a path homotopy between f and g, so $[f] = [g]$. □

Theorem 22. *For all $x \in S^1$*

$$\pi_1(S^1, x) \cong \mathbb{Z}.$$

Proof. Since S^1 is path-connected we can work with $x = \rho(0)$ where $\rho : \mathbb{R} \longrightarrow S^1$ is the canonical covering map. Then $\rho^{-1}(x) = \mathbb{Z}$ and thus since $\mathbb{R}$ is simply connected we see that the lifting correspondence

$$\pi_1(S^1, x) \longrightarrow \mathbb{Z}$$

is a bijection. Our goal is to simply now show that this is a homomorphism.

Given $[f], [g] \in \pi_1(S^1, x)$ let $\overline{f}$ and $\overline{g}$ be their lifts beginning at 0. Our goal is to show that $\overline{f \cdot g}(1) = \overline{f}(1) + \overline{g}(1)$. To this extent, we need to compute $\overline{f \cdot g}$. Let $\tilde{g} = \overline{f}(1) + \overline{g}$. Becuase $\rho(\overline{f}(1) + \overline{g}) = \rho(\overline{g}) = g$ it follows that $\tilde{g}$ is a lifting of g beginning at $\overline{f}(1)$. Then $\overline{f} \cdot \tilde{g}$ is a path in $\mathbb{R}$ starting at 0 and further $\rho \circ (\overline{f} \cdot \tilde{g}) = (\rho \circ \overline{f}) \cdot (\rho \circ \tilde{g}) = f \cdot g$ is a lifting of $f \cdot g$ so this shows that $\overline{f \cdot g} = \overline{f} \cdot \tilde{g}$. It is routine to verify $(\overline{f} \cdot \tilde{g})(1) = \overline{f}(1) + \overline{g}(1)$. Thus we have shown that this map is a homomorphism. □